UNIVERSITY OF IOANNINA

DEPARTMENT OF ECONOMICS

M.Sc. in Economics

MICROECONOMIC THEORY I

Teaching: A. Lapatinas

Problem Set II

*(Note: The number of * indicates exercise's difficulty level)*

1. (*)True or false? If $V(y)$ is a convex set, then the associated production set Y must be convex.

2. (*)Prove that convex input requirement set is equivalent to quasi-concave production function.

3. (*) (a) What is the elasticity of substitution for the general CES technology $y=(a_1x_1^{\rho}+a_2x_2^{\rho})^{1/\rho}$ when $a_1 \neq a_2$?

   (b) Show that we can always write the CES technology in the form $f(x_1,x_2)=A(\rho)[bx_1^{\rho}+(1-b)x_2^{\rho}]^{1/\rho}$.

4. (*) (a) Define the output elasticity of a factor i to be

$$\varepsilon_i(x)=\frac{\partial f(x)}{\partial x_i}\frac{x_i}{f(x)}$$

   What is the output elasticity of each factor of the production functions: $y=k(1+x_1^{-a}x_2^{-\beta})^{-1}$, $y=x_1^{a}x_2^{\beta}$, $y=(x_1^{\rho}+x_2^{\rho})^{1/\rho}$?

   (b) If $\varepsilon(x)$ is the elasticity of scale and $\varepsilon_i(x)$ is the output elasticity of factor i, show that $\varepsilon(x)=\sum_{i=1}^{n}\varepsilon_i(x)$. What is the elasticity of scale of each of the above technologies?

5. (*) If $f(x)$ is a homothetic technology and x and x' produce the same level of output, then tx and tx' must also produce the same level of output. Can you prove this rigorously?

6. (*)Show that a profit maximizing bundle will typically not exist for a technology that exhibits increasing returns to scale as long as there is some point that yields a positive profit.

7. (*) For each input requirement set determine if it is regular, monotonic, and/or convex. Assume that the parameters a and b and the output levels are strictly positive.
   (a) $V(y) = \{x_1, x_2 : ax_1 \geq \log y, bx_2 \geq \log y\}$
   (b) $V(y) = \{x_1, x_2 : ax_1 + bx_2 \geq y, x_1 > 0\}$
   (c) $V(y) = \{x_1, x_2 : ax_1 + \sqrt{x_1 x_2} + bx_2 \geq y\}$
   (d) $V(y) = \{x_1, x_2 : ax_1 + bx_2 \geq y\}$
   (e) $V(y) = \{x_1, x_2 : x_1(1-y) \geq a, x_2(1-y) \geq b\}$
   (f) $V(y) = \{x_1, x_2 : ax_1 - \sqrt{x_1 x_2} + bx_2 \geq y\}$
   (g) $V(y) = \{x_1, x_2 : x_1 + \min(x_1, x_2) \geq 3y\}$

8. (*) Calculate explicitly the profit function for the technology $y = x^a$, for $0 < a < 1$ and verify that it is homogeneous of degree 1 and convex in (p, w), where p and w are output and input price respectively.

9. (*) Let $f(x_1, x_2)$ be a production function with two factors and let w_1 and w_2 be their respective prices. Show that the elasticity of the factor share $(w_2 x_2 / w_1 x_1)$ with respect to (x_2 / x_1) is given by $1 - 1/\sigma$ and with respect to (w_2 / w_1) is given by $1 - \sigma$, where σ is the elasticity of substitution.

10. (*) The production function is $f(x) = 20x - x^2$ and the price of output is normalized to 1. Let w be the price of the input $x \geq 0$.

(a) What is the first order condition for profit maximization if $x > 0$?

(b) For what values of w will the optimal x be zero?

(c) For what values of w will the optimal x be 10?

(d) What is the factor demand function?

(e) What is the profit function and what is its derivative with respect to w?

11. (**) A competitive profit maximizing firm has a profit function $\pi(w_1, w_2) = \varphi_1(w_1) + \varphi_2(w_2)$. The price of output is normalized to 1.

(a) What do we know about the first and second derivatives of the functions $\varphi_i(w_i)$?

(b) If $x_i(w_1, w_2)$ is the factor demand function for factor i, what is the sign of $\partial x_i / \partial w_j$?

(c) Let $f(x_1, x_2)$ be the production function that generated the profit function of this form. What can we say about the form of this production function?

12. (*) Consider the technology described by $y = 0$ for $x \le 1$ and $y = \ln x$ for $x > 1$. Calculate the profit function for this technology.

13. (*) (a) Given the production function $f(x_1, x_2) = a_1 \ln x_1 + a_2 \ln x_2$, calculate the profit maximizing demand and supply functions, and the profit function. Assume an interior solution and $a_i > 0$.

(b) Do the same for the production function $f(x_1, x_2) = x_1^{a_1} x_2^{a_2}$. Assume $a_i > 0$. What restrictions must a_1 and a_2 satisfy?

(c) Do the same for the production function $f(x_1, x_2) = \min\{x_1, x_2\}^a$. What restrictions must a satisfy?

14. (**) A price-taking firm produces output q from inputs z_1 and z_2 according to a differentiable concave production function $f(z_1, z_2)$. The price of its output is $p > 0$ and the prices of its inputs are $(w_1, w_2) >> 0$. However, there are two unusual things about this firm. First, rather than maximizing profit, the firm maximizes revenue (the manager wants her firm to have bigger dollar sales than any other). Second, the firm is cash constrained. In particular, it has only C dollars on hand before production and, as result, its total expenditures on inputs cannot exceed C.

    Suppose one of your econometrician friends tells you that she has used repeated observations of the firm's revenues under various output prices, input prices, and levels of the financial constraint and has determined that the firm's revenue level R can be expressed as the following function of the variables (p, w_1, w_2, C):

$$R(p, w_1, w_2, C) = p\left[\gamma + \ln C - a \ln w_1 - (1-a)\ln w_2\right]$$

    (γ and a are scalars whose values she tells you) What is the firm's use of inputs z_1 and z_2 when prices are (p, w_1, w_2) and it has C dollars of cash on hand?

15. (**) Prove rigorously that profit maximization implies cost minimization.

16. (**) A firm has two plants with cost functions $c_1(y_1) = \frac{y_1^2}{2}$ and $c_2(y_2) = y_2$. What is the cost function for the firm?

17. (**) A firm has two plants. One plant produces output according to the production function $x_1^a x_2^{1-a}$. The other plant has a production function $x_1^b x_2^{1-b}$. What is the cost function for the firm?

18. (**) (a) Show that if the production function is homogeneous of degree n, then the cost function can be written $c(w,y) = y^{\frac{1}{n}} b(w)$, where $w \in \mathbb{R}^n_{++}$.

    (b) Show that if the production function is homothetic, then $\varepsilon_{cy} = \varepsilon^{-1}$, where ε_{cy} is the cost elasticity with respect to output and ε is the elasticity of scale.

19. (*) A firm has a production function $y = x_1 x_2$. If the minimum cost of production at $w_1 = w_2 = 1$ is equal to 4, what is y equal to?

20. (*) Show that $\dfrac{\partial x_i(w,y)}{\partial y} > 0$ if and only if marginal cost at y is increasing in w_i.

21. (*) A firm produces output y in a competitive market using a cost function $c(y)$ which exhibits increasing marginal costs. Of the output, a fraction $1-a$ is defective and cannot be sold. If the output price is p
    (a) Calculate the derivative of profits with respect to a and its sign.
    (b) Calculate the derivative of output with respect to a and its sign.
    (c) Suppose that there are N identical producers, let $D(p)$ be the demand function and let $p(a)$ be the competitive equilibrium price. Calculate $(dp/da)/p$ and its sign.

22. (*) Consider a profit maximizing firm that produces a good which is sold in a competitive market. It is observed that when the price of the output good rises, the firm hires more skilled workers but fewer unskilled workers. Now the unskilled workers unionize and succeed in getting their wage increased. Assume that all other prices remain constant.
    (a) What will happen to the firm's demand for unskilled workers?
    (b) What will happen to the firm's supply of output?

23. (*) For each cost function determine if it is homogeneous of degree one, monotonic, concave and continuous. If it is derive the associated production function:

(a) $c(w,y) = y^{1/2}(w_1 w_2)^{3/4}$

(b) $c(w,y) = y(w_1 + \sqrt{w_1 w_2} + w_2)$

(c) $c(w,y) = y(w_1 e^{-w_1} + w_2)$

(d) $c(w,y) = y(w_1 - \sqrt{w_1 w_2} + w_2)$

(e) $c(w,y) = (y + \frac{1}{y})\sqrt{w_1 w_2}$

24. (*) A firm's cost function is $c(w_1, w_2, y) = w_1^a w_2^\beta y$. What can we say about a and β?

25. (*) (a) Show that the cost function of a Leontief technology, $y = \min\{\frac{x_1}{\beta_1}, \frac{x_2}{\beta_2}\}$, is given by $c(w,y) = y(\beta_1 w_1 + \beta_2 w_2)$.

(b) Show that the cost function of a linear technology, $y = a_1 x_1 + a_2 x_2$, is given by $c(w,y) = y \min\left\{\frac{w_1}{a_1}, \frac{w_2}{a_2}\right\}$.

(c) Show that the cost function of a Cobb-Douglas technology, $y = A x_1^a x_2^{1-a}$, is given by $c(w,y) = y w_1^a w_2^\beta B$, where B depends only on A and a.

For all the above cases draw the cost minimization problem. Find the conditional factor demand functions. Draw total, average and marginal cost functions.

26. (**) Suppose the strictly quasi-concave production function $f(x_1, x_2)$ and suppose factor prices w_1, w_2. Suppose also that profit function is strictly concave *with respect to output*. Normalize output price to 1.

(a) Solve profit maximization problem and find factor demand functions $x_i^*(w_1, w_2)$ and supply function $y^*(w_1, w_2)$.

(b) Solve cost minimization problem and find the conditional factor demand functions $\hat{x}_i(w, y)$. Show that for $y = y^*$, it is $\hat{x}_i = x_i^*$.

(c) Alternatively, take the cost function $c(w, y)$. Solve the maximization problem $\Pi = R(y) - c(y)$ and find $\hat{y}(w_1, w_2)$. Show that $\hat{y}(w_1, w_2) = y^*(w_1, w_2)$.